

THE GRAPH TOPOLOGY

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ABSTRACT. We study topological properties of the graph topology on the space of continuous real-valued functions.

1. PRELIMINARIES

Let X be a topological space and $C(X)$ be the space of continuous real-valued functions. The graph topology τ_Γ was introduced by Naimpally [Na] and has as its basic open sets, sets of the form $\{f \in C(X) : f \subset G\}$, where G is an open subset of $X \times R$ (here f is identified with its graph). If G is an open set in $X \times R$ denote by F_G the set $\{f \in C(X) : f \subset G\}$. It is known that for a T_1 space X the graph topology on $C(X)$ coincides with the Vietoris topology on $C(X)$.

Najprv by sme asi mali uvazovat graph topology na vseobecnych X, Y .

Let X and Y be topological spaces. Denote the graph topology on $C(X, Y)$, the space of continuous functions from X to Y , by τ_Γ .

If X is a T_1 space then the Vietoris topology and the graph topology on $C(X, Y)$ coincide:

(For every $x \in X$ and U open in Y , let $W(x, U) = \{f \in C(X, Y) : f(x) \in U\}$. Since X is T_1 , the set $V = (X \times U) \cup (X \setminus \{x\}) \times Y$ is open in $X \times Y$ and clearly $F_V = W(x, U)$. Now let $G \times U$ be a product of open sets in X and Y , respectively and consider the set $(G \times U)^-$. It is easy to verify that

$$(G \times U)^- = \cup_{x \in G} W(x, U).$$

The separation axioms T_1 and T_2 on $(C(X, Y), \tau_\Gamma)$ are solved in [Na]. In what follows let X, Y be Hausdorff spaces. It is easy to verify that if $X \times Y$ is a normal space, then $(C(X, Y), \tau_\Gamma)$ is regular. Let G be an open set in $X \times Y$ and $f \in C(X, Y)$

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such that $f \in F_G$. Since the graph of f is a closed set in $X \times Y$ and $f \subset G$, the normality of $X \times Y$ implies that there is an open set H in $X \times Y$ such that

$$f \subset H \subset \overline{H} \subset G.$$

Since the set $F_{\overline{H}} = \{h \in C(X, Y) : h \subset \overline{H}\}$ is closed in $(C(X, Y), \tau_\Gamma)$, we are done.

The condition of normality of $X \times Y$ is essential.

Example. Let W_0 be the ordinals less than or equal to the first uncountable ordinal Ω with the order topology and $W = W_0 \setminus \{\Omega\}$, with the induced topology from W_0 . It is known that $W \times W_0$ is not normal. We will show that $(C(W, W_0), \tau_\Gamma)$ is not regular. Let $f : W \rightarrow W_0$ be the identity function. The set $W \times W$ is open in $W \times W_0$ and $f \subset W \times W$. There is no open set G in $W \times W$ such that

$$f \subset G \subset \overline{G} \subset W \times W.$$

It is easy to verify that for every open set $G \subset W \times W$ with $f \in F_G$ we have that the closure of F_G in the graph topology is not contained in $F_{W \times W}$. Let G be an open set in $W \times W_0$ such that

$$f \subset G \subset W \times W.$$

There is $\alpha \in W$ such that $(y, z) \in G$, if $y > \alpha$, $z > \alpha$. Define a function g as follows: $g(x) = x$ for $x \leq \alpha + 1$ and $g(x) = \Omega$ for every $x > \alpha + 1$. It is easy to verify that the function g is in the closure of F_G in the graph topology and of course $g \notin F_{W \times W}$.

Thus $(C(W, W_0), \tau_\Gamma)$ is not regular.

Denote further by $LSC(X, (0, 1))$ the space of all lower semicontinuous functions defined on X with values in the open interval $(0, 1)$.

Lemma 1.1. Let X be a topological space. $(C(X), \tau_\Gamma)$ has as a base all sets of the form

$$B(f, \epsilon) = \{g \in C(X) : |f(x) - g(x)| < \epsilon(x)\},$$

where $f \in C(X)$ and $\epsilon \in LSC(X, (0, 1))$.

Proof. Let G be an open set in $X \times R$ with $f \subset G$. It is easy to verify that there is an open set U in $X \times R$ such that $f \subset U \subset G$, U is locally bounded and $U(x)$ is an open interval for every $x \in X$. For every $x \in X$ put

$$\epsilon(x) = \min\{|f(x) - y| : y \in U(x)^c\}.$$

It is easy to verify that ϵ is a positive lower semicontinuous function. Put $\eta(x) = \min\{\epsilon(x), 1/2\}$. Then $\eta \in LSC(X, (0, 1))$ and of course $B(f, \eta) \subset U \subset G$.

Lemma 1.1'. *Let X be a topological space and (Y, d) be a metric space. Then $(C(X, Y), \tau_\Gamma)$ has as a base all sets of the form*

$$B(f, \epsilon) = \{g \in C(X, Y) : d(f(x), g(x)) < \epsilon(x), \forall x \in X\},$$

where $f \in C(X, Y)$ and $\epsilon \in LSC(X, (0, 1))$.

Proof. Let V be an open set in $X \times Y$ with $f \subset V$. For every $x \in X$ put

$$V(x) = \{y \in Y : (x, y) \in V\}.$$

Without loss of generality we can suppose that $V(x)^c$ is nonempty for every $x \in X$. Now define the functions σ, η as follows:

$$\sigma(x) = d(f(x), V(x)^c) \text{ for every } x \in X, \text{ and}$$

$$\eta(x) = \sup\{\inf\{\sigma(z) : z \in U\} : U \in \mathcal{U}(x)\} \text{ for every } x \in X,$$

where $\mathcal{U}(x)$ is the neighborhood base at x . Put $\epsilon(x) = \min\{\eta(x), 1\}$ for every $x \in X$. It is easy to verify that ϵ is lower semicontinuous and positive and $B(f, \epsilon) \subset F_V$.

We need to prove that $B(f, \epsilon) \in \tau_\Gamma$ for $f \in C(X, Y)$ and $\epsilon \in LSC(X, (0, 1))$. We prove that the set

$$H(f, \epsilon) = \{(x, t) \in X \times Y : d(f(x), t) < \epsilon(x), x \in X\}$$

is open in $X \times Y$. Let $(x, t) \in H(f, \epsilon)$. Put $\alpha(x) = \epsilon(x) - d(f(x), t)$. There is an open neighborhood O_x of x such that $d(f(x), f(z)) < \alpha(x)/3$ and $\epsilon(z) > \epsilon(x) - \alpha(x)/3$ for every $z \in O_x$. We claim that

$$H = O_x \times \{y \in Y : d(y, t) < \alpha(x)/3\} \subset H(f, \epsilon).$$

Let $(z, y) \in H$. Then $d(f(z), y) \leq d(f(z), f(x)) + d(f(x), y) \leq d(f(z), f(x)) + d(f(x), t) + d(t, y) \leq \alpha(x)/3 + d(f(x), t) + \alpha(x)/3 = 2\alpha(x)/3 + d(f(x), t) = 2/3(\epsilon(x) - d(f(x), t)) + d(f(x), t) = 2\epsilon(x)/3 + d(f(x), t)/3 < \epsilon(z)$. Thus $d(f(z), y) < \epsilon(z)$, so we are done.

Proposition 1.2. *Let X be a topological space. The following are equivalent:*

- (1) $\tau_\omega = \tau_\Gamma$ on $C(X)$;
- (2) for every $\epsilon \in LSC(X, (0, 1))$ there is $\eta \in C(X, (0, 1))$ such that $\eta(x) \leq \epsilon(x)$ for every $x \in X$.

Of course, if X is countably compact or countably paracompact normal space, then the property (2) in Proposition 1.2 is satisfied. It would be interesting to know whether the property (2) characterizes countable paracompactness in the class of normal spaces.

If X is a T_1 -space, then also $(C(X), \tau_\Gamma)$ is a T_1 -space. By Lemma 1.1 we know that $(C(X), \tau_\Gamma)$ is a topological group. Thus if X is a T_1 -space $(C(X), \tau_\Gamma)$ is

a Tychonoff space. $(C(X), \tau_\Gamma)$ is a topological vector space if and only if X is countably compact.

The family $\{B(\epsilon) : \epsilon \in LSC(X, (0, 1))\}$ of sets of the form $B(\epsilon) = \{(f, g) \in C(X) \times C(X) : |f(x) - g(x)| < \epsilon(x), \forall x \in X\}$ clearly satisfies the conditions for a base for a uniformity on $C(X)$ ([Ke]). Since obviously, $B(\epsilon)[f] = B(f, \epsilon)$, the uniformity induces the graph topology τ_Γ on $C(X)$. This uniformity will be called the uniformity of the graph topology and will be denoted by \mathcal{U}_Γ .

Proposition 1.3. *Let X be a topological space. $(C(X), \mathcal{U}_\Gamma)$ is a complete uniform space.*

Proof. Let $\{f_\lambda : \lambda \in \Lambda\}$ be a \mathcal{U}_Γ -Cauchy net in $C(X)$. Since \mathcal{U}_Γ contains the uniformity \mathcal{U}_u of uniform convergence, the net is also \mathcal{U}_u -Cauchy and therefore is uniformly convergent to some $f \in C(X)$. The rest of the proof is similar to the proof of an analogous statement for \mathcal{U}_u . For $\epsilon \in LSC(X, (0, 1))$ there is $\lambda_0 \in \Lambda$ such that for each $\lambda_1 \geq \lambda_0$, each $\lambda_2 \geq \lambda_0$ and each $x \in X$ we have

$$|f_{\lambda_1}(x) - f_{\lambda_2}(x)| < \epsilon(x)/2.$$

Fix $\lambda \geq \lambda_0$ and $x \in X$. There is $\eta \geq \lambda$ such that $|f_\eta(x) - f(x)| < \epsilon(x)/2$. Thus we have

$$|f_\lambda(x) - f(x)| \leq |f_\lambda(x) - f_\eta(x)| + |f_\eta(x) - f(x)| < \epsilon(x).$$

Consequently $f_\lambda \in B(f, \epsilon)$ for all $\lambda \geq \lambda_0$, so the net $\{f_\lambda : \lambda \in \Lambda\}$ τ_Γ -converges to f , which completes the proof.

2. TOPOLOGICAL PROPERTIES OF THE GRAPH TOPOLOGY

Proposition 2.1. *Let X be a Tychonoff space. The following are equivalent:*

- (1) $(C(X), \tau_\Gamma)$ is first countable;
- (2) $(C(X), \tau_\Gamma)$ is metrizable;
- (3) $(C(X), \tau_\Gamma)$ is completely metrizable;
- (4) $(C(X), \tau_\Gamma)$ is Čech-complete;
- (5) X is countably compact;
- (6) $(C(X), \tau_\Gamma)$ is Frechet;
- (7) $(C(X), \tau_\Gamma)$ has a countable tightness.

Proof. Since the equivalence $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5)$ is proved in [HH], it is sufficient to prove that $(7) \Rightarrow (5)$. Suppose X is not countably compact. There is an infinite set $\{x_n : n \in \omega\}$ in X without an accumulation point. It is easy to verify that there is a sequence of open neighborhoods $\{O(x_n) : n \in \omega\}$ of the points $\{x_n : n \in \omega\}$ such that the sequence $\{\overline{O(x_n)} : n \in \omega\}$ is pairwise disjoint.

Let f_0 be the function identically equal to 0. Then f_0 belongs to the τ_Γ -closure of the set

$$L = \{g \in C(X) : g(x_n) > 0, \forall n \in \omega\}.$$

Indeed, let $\epsilon \in LSC(X, (0, 1))$ and consider the set $B(f_0, \epsilon)$. Put

$$V(f_0, \epsilon) = \{(x, t) : -\epsilon(x) < t < \epsilon(x)\}.$$

Then $V(f_0, \epsilon)$ is an open set in $X \times R$. For every $n \in \omega$, there is an open set $U(x_n) \subset O(x_n)$ in X with $x_n \in U(x_n)$ and an open interval (a_n, b_n) containing 0 with

$$U(x_n) \times (a_n, b_n) \subseteq V(f_0, \epsilon).$$

For every $n \in \omega$, let $H(x_n)$ be an open set in X such that $x_n \in H(x_n) \subset \overline{H(x_n)} \subset U(x_n)$ and let $h_n : \overline{H(x_n)} \rightarrow [0, b_n/n]$ be a continuous function such that $h_n(x_n) = b_n/n$ and $h_n(z) = 0$ for every $z \in \overline{H(x_n)} \setminus H(x_n)$. Now let $h : X \rightarrow [0, 1]$ be defined as follows: $h(x) = h_n(x)$ for every $x \in \overline{H(x_n)}$ and $h(x) = 0$ otherwise. It is easy to verify that h is continuous function and $h \in B(f_0, \epsilon)$.

The countable tightness of $(C(X), \tau_\Gamma)$ implies that there is a countable subfamily $L' = \{f_n : n \in \omega\}$ of L such that f_0 is in the τ_Γ -closure of L' . Let η be a positive function defined on X as follows: $\eta(x_n) = f_n(x_n)/(n+1)$ for every $n \in \omega$ and $\eta(x) = 1/2$ otherwise. It is easy to verify that $\eta \in LSC(X, (0, 1))$.

Since $f_n \notin B(f_0, \eta)$ for every $n \in \omega$, we have a contradiction.

Myslim, ze plati aj nasledujuce:

Proposition 2.1'. *Let X be a Tychonoff space and Y be a metrizable space which contains an arc. The following are equivalent:*

- (1) $(C(X, Y), \tau_\Gamma)$ is metrizable;
- (2) $(C(X, Y), \tau_\Gamma)$ is first countable;
- (3) $(C(X, Y), \tau_\Gamma)$ is Frechet;
- (4) $(C(X, Y), \tau_\Gamma)$ has a countable tightness;
- (5) X is countably compact.

Proposition 2.2. *Let X be a Tychonoff space. The following are equivalent:*

- (1) $(C(X), \tau_\Gamma)$ is second countable;
- (2) $(C(X), \tau_\Gamma)$ has a countable network;
- (3) $(C(X), \tau_\Gamma)$ is separable;
- (4) $(C(X), \tau_\Gamma)$ has a countable chain condition;
- (5) X is compact and metrizable.

Proof. (1) \Rightarrow (2), (2) \Rightarrow (3) and (3) \Rightarrow (4) are trivial. (4) \Rightarrow (5) The uniform topology τ_u is weaker than the graph topology τ_Γ on $C(X)$. Thus also $(C(X), \tau_u)$ has a countable chain condition. By [MN] X must be compact and metrizable.

(5) \Rightarrow (1) If X is compact and metrizable, the graph topology τ_Γ on $C(X)$ coincides with the topology τ_u of uniform convergence, which is second countable if X is compact and metrizable.

Proposition 2.2'. *Let X be a Tychonoff space and Y be a metrizable space. The following are equivalent:*

- (1) $(C(X, Y), \tau_\Gamma)$ is second countable;
- (2) $(C(X, Y), \tau_\Gamma)$ has a countable network;
- (3) $(C(X, Y), \tau_\Gamma)$ is separable;
- (4) X is compact and metrizable and Y is separable.

Proof. (1) \Rightarrow (2) and (2) \Rightarrow (3) are trivial. (3) \Rightarrow (4) The uniform topology τ_u is weaker than the graph topology τ_Γ on $C(X, Y)$. Thus also $(C(X, Y), \tau_u)$ is separable. It is the well known fact that X must be compact and metrizable and Y must be separable.

(4) \Rightarrow (1) If X is compact and metrizable and Y is separable and metrizable, then $(C(X, Y), \tau_u)$ is separable and metrizable. Thus $(C(X, Y), \tau)$ is second countable and $\tau_u = \tau_\Gamma$.

Proposition 2.3. *Let X be a topological space. Then $(C(X), \tau_\Gamma)$ is a Baire space.*

Proof. Let $\{G_n : n \in \omega\}$ be a sequence of open dense sets in $(C(X), \tau_\Gamma)$. W.l.o.g we can suppose that $G_{n+1} \subset G_n$ for every $n \in \omega$. We want to prove that $\bigcap_{n \in \omega} G_n$ is a dense set in $(C(X), \tau_\Gamma)$. Let U be an open set in $(C(X), \tau_\Gamma)$. Define a sequence $\{f_n : n \in \omega\}$ in $C(X)$ and a sequence $\{\epsilon_n : n \in \omega\}$ in $LSC(X, (0, 1))$ such that

$$B(f_0, \epsilon_0) \subset U \cap G_0, B(f_n, \epsilon_n) \subset B(f_{n-1}, \epsilon_{n-1}/2) \cap G_n \text{ for every } n \geq 1$$

and $\epsilon_n < 1/2^n$ for every $n \in \omega$. There is a continuous function f such that $\{f_n : n \in \omega\}$ τ_u -converges to f . We will show that $f \in U \cap \bigcap_{n \in \omega} G_n$.

Let $n \in \omega$. We show that $f \in B(f_n, \epsilon_n)$. For every $m > n$ we have $f_m \in B(f_n, \epsilon_n/2)$. Let $x \in X$. The τ_u -convergence of $\{f_n : n \in \omega\}$ to f implies that there is $n_1 > n$ such that $|f_m(x) - f(x)| < \epsilon_n(x)/2$ for every $m \geq n_1$. We have

$$|f_n(x) - f(x)| \leq |f_n(x) - f_{n_1}(x)| + |f_{n_1}(x) - f(x)| < \epsilon_n(x)/2 + \epsilon_n(x)/2.$$

Thus $f \in B(f_n, \epsilon_n)$ and $B(f_n, \epsilon_n) \subset G_n$ for every $n \in \omega$; i.e. $f \in U \cap \bigcap_{n \in \omega} G_n$.

Proposition 2.4. *Let X be a T_1 topological space. Let Q be a subset of $C(X)$. The following are equivalent:*

- (1) Q is compact in $(C(X), \tau_\Gamma)$;
- (2) Q is countably compact in $(C(X), \tau_\Gamma)$;
- (3) Q is pseudocompact in $(C(X), \tau_\Gamma)$.

Proof. (1) \Rightarrow (2) and (2) \Rightarrow (3) are trivial. We prove (3) \Rightarrow (1). If Q is pseudocompact in $(C(X), \tau_\Gamma)$, the uniform space (Q, \mathcal{U}_Γ) is totally bounded [En]. Moreover, $(C(X), \mathcal{U}_\Gamma)$ is complete by Proposition 1.3. If Q is closed in $(C(X), \tau_\Gamma)$, then the uniform space (Q, \mathcal{U}_Γ) is complete and totally bounded and hence compact. So it suffices to show that Q is closed in $(C(X), \tau_\Gamma)$.

If (Q, τ_Γ) is pseudocompact, then so is (Q, τ_u) , because any τ_u -continuous function $f : Q \rightarrow R$ is τ_Γ -continuous and therefore bounded. Thus (Q, τ_u) is pseudocompact and metrizable and hence compact. So Q is closed in $(C(X), \tau_u)$. Thus Q is closed also in $(C(X), \tau_\Gamma)$. This completes the proof.

3. CARDINAL INVARIANTS

Theorem 3.1. *Let X be a Tychonoff space. Then for every closed discrete set D in X we have $|D| \leq \chi(C(X))$, where $|D|$ is the cardinality of D .*

Proof.

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